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NEW METHOD FOR FINDING THE WEIGHT DISTRIBUTION AND SPECTRUM BY TESTING THE OPTIMAL SOLUTION OF THE POINTS IN PG(3,2)

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ABSTRACT

In this paper, the basic solution was found by using Vogel's approximation method for the projective space PG(3,2) when m_i=5,7,15 by converting the points and lines of PG(3,2) into a transportation problem with five rows and seven columns. The problem was balanced and treating Degeneracy. Then we tested the optimality of the basic solution using the modified distribution method. And then we construct a new linear codes of dimension p=4 and smallest length m_i for which a [m_i, p, d]-codes exists, and by a geometric method we found the weight distribution and the spectrum of a linear code [m_i, p, d] over a field F_2.

Keywords: *Vogel's Approximation, Projective Space*, *Linear Codes, Weight Distribution, Spectrum.*

1. INTRODUCTION

Suppose there are many important messages to be sent through a noisy communication channel, in order to give these messages some protection against error on the channel, they are encoded in to codewords. The set consisting of these codewords is called a code[1]. Let m,p,d be positive integers. A code C: [m,p,d] over Galois Field (GF(q)) is a p-dimensional subspace of $(GF(q))^n$ with minimum distance d. In a code C of length m let $\mathbb{C}_i$ denote the number of codewords with Hamming weight i. The weight enumerator of C is defined by $1+\mathbb{C}_1z+\mathbb{C}_2z^2+\cdots+\mathbb{C}_mz_m$. The sequence $(1,\mathbb{C}_1, \mathbb{C}_2, \cdots, \mathbb{C}_m)$ is called the weight distribution of the code C. In this research we study more constructions of codes from an optimal Value for the code C:[m_i,4] over F_2,when m_i=5,7,15. The transportation problem can be stated mathematically as a linear programming problem as:

Minimize: $z=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}$

Subject to: $\sum_{j=1}^{n}x_{ij}\le a_i; i=1,2,\dots,m$

$$\sum_{i=1}^{m}x_{ij}\ge b_j;\ j=1,2,\dots,n$$

$x_{ij}\ge 0$ for all i and j.

Such that, x_{ij} is the amount shipped.

c_{ij} is the transportation cost per unit.

a_i is the amount of supply at source i.

b_i is the amount of demand at destination j.

There are Two types of the Transportation Problems:

a. Balancing the transportation problem: assume that total availability is equal to the total requirement:

$$\sum_{i=1}^{m}a_i=\sum_{j=1}^{n}b_j.$$

b. Unbalancing the transportation problem: assume that total availability is not equal to the total requirement:

$$\sum_{i=1}^{m}a_i\ne\sum_{j=1}^{n}b_j.$$

The problem is treatment before starting the solution, that is, making it balanced by adding an imaginary row or imaginary column to the transportation matrix with a cost equal to zero, so that the total supply equals the total total demand. The objective of the transportation Problems is to determine the unknowns x_{ij} that will minimize the total transportation cost while satisfying the supply and demand restrictions[2], [3], [4].

2. PRELIMINARIES

2.1 Definition

Let F_q^m denotes the vector space of m-tuple. A (m,M)-code C over Galois field F_q is a subset of F_q^m of size M. A linear $[m, p]_q$-code over F_q is a p-dimensional subspace of F_q^m and size $M=q^p$. The vectors in the linear code C are called codewords and them by $w=w_1\ w_2\dots w_m$ where $w_i \in F_q$. In other words, a q-ary linear code C of length m and dimension p, or a $[m, p]_q$ -code, is a p-dimensional subspace of F_q^m, where p=4. Every subspace of C is referred to as a sub code of C.
The inner product of two vectors $w=(w_1,w_2,\dots,w_m)$ and $y=(y_1,y_2,\dots,y_m)$ from F_q^m is denoted by $w_y=w_1y_1+w_2y_2+\cdots+w_my_m$. Two vectors are said to be orthogonal if their inner product is 0. The set of all vectors of F_q^m orthogonal to all codewords from C is called the orthogonal code: $C^{\perp}=\{yw\in F_q^m \mid wy=0 \text{ for any } y\in C\}$. By a well-known fact from linear algebra, the code $C^{\perp}$ is a linear $[m,m-p]_q$-code. If C is a linear code that, as a vector space over F_q^m has dimension p, then we say that C is a [m,p] linear code over F_q[5], [6].

2.2 Definition

A code C is called a [m,p,d]-code if d is the minimum nonzero weight in C. Let $\mathbb{G}_i$ be the number of codewords of weight i in C. The weight enumerator of C is defined by $1+\mathbb{G}_1w+\mathbb{G}_2w^2+\cdots+\mathbb{G}_mw^m$. The weight distribution (W.D.) of C is the list of numbers $A_i=|\{w\in C \mid W(w)=i\}|$. The weight distribution with $(\mathbb{G}_0, \mathbb{G}_d,\dots, \mathbb{G}_i,\dots)=(1,\alpha,\dots,w,\dots)$ is also expressed as $0^1,d^{\alpha},\dots,i^{w}\dots$. And the list $\mathbb{G}_i(C)$ for $0 \le i \le m$ is called the weight spectrum of C[7], [8].

2.3 Theorem

Let C be a [m,p,d]-code over F_q then

1. $\mathbb{G}_0(C)+\mathbb{G}_1(C)+\dots+\mathbb{G}_m(C)=q^p$
2. $\mathbb{G}_0(C)=1$ and $\mathbb{G}_1(C)=\mathbb{G}_2(C)=\dots=\mathbb{G}_{d-1}(C)=0$[9].

2.4 Theorem

For every linear [m,p,d]-code C, $0 < \mathbb{G}_1(C) < \mathbb{G}_2(C) < \dots < \mathbb{G}_p(C) \le m$[1].

2.5 Definition

A generator matrix G of a [m,p]-code C is a (p×m) matrix whose rows form a basis for C.

$$G=\begin{bmatrix} g_0 \\ g_1 \\ . \\ . \\ . \\ g_{p-1} \end{bmatrix}=\begin{bmatrix} g_{0,0} & g_{0,1} & . & . & g_{0,m-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ g_{p-1,0} & g_{p-1,1} & . & . & g_{p-1,m-1} \end{bmatrix}$$

,Such that $G^*=U_{(1\times p)}G_{(p\times m)}$,(U is a different row vectors)[5], [10].

3. FINDING THE INITIAL BASIC ACCEPTABLE SOLUTION FOR THE TRANSPORTATION PROBLEMS

The basic acceptable solution to the balanced transportation problem was found in this research using Vogel's method

3.1 Vogel Approximation Method

It is considered one of the most important methods for the basic and acceptable initial solution. It includes calculating the difference between the two lowest costs in each row or column, then choosing the larger difference for the rows and columns, and then choosing the square that contains the lowest cost in that area. The row or column that was previously determined, and the largest available quantity is allocated to meet the final needs or the originally available quantities run out, then the row or column that is achieved is deleted[11].

4. TEST THE OPTIMAL SOLUTION FOR THE TRANSPORTATION PROBLEM

Obtaining the basic, acceptable initial solution does not mean the end of the problem. Rather, we must use other methods to test whether the basic solution obtained from applying Vogel's method is the optimal solution (the only solution), that is, it is not possible to find a better solution, or whether there are solutions. Optimum, and to test the optimality of the accepted basic solution, it is necessary to ensure that the following condition is satisfy: Number of squares occupied in the initial solution = number of rows (m) + number of columns (n) - (1) … 1
If the above equation is not satisfied, then this phenomenon is called the phenomenon of Degeneracy and must be treated because it does not allow us to form closed paths. Therefore, we add the value of ∈ to one of the non-essential variables

(an empty cell) with the lowest cost, so that the value of ∈ is very small and larger than zero, as the result of subtracting it from any number or adding it with any number equals the number itself, and then we find the optimal solution using the modified distribution method[12].

4.1 Modified Distribution Method

This method is based on the binary property of formulating the linear program for the transportation problem, as this property says that there is a set of u_i values for each row and a set of v_j values for each column for each of the non-empty cells, as these values are found according to the following equations:

$$c_{ij} = u_i + v_j \ldots(1)$$

Each empty cell has a rating calculated as follows:

$$d_{ij} = c_{ij} - u_i - v_j \ldots(2).$$

Since: c_{ij}: It is the transportation cost of non-empty cells.

d_{ij}: It is the result of evaluating empty cells.

u_i: Evaluation values for columns.

v_j: Evaluation values for rows.

5. TRANSFORMING THE STUDIED ISSUE INTO A TRANSPORTATION PROBLEM AND FINDING THE OPTIMAL SOLUTION FOR IT

The transportation problem was applied to the three-dimensional projective space PG(3,2) when m_i=5,7,15; i=1,2,3 by converting the levels of the projective space into three transportation problems, each problem consisting of five rows and seven columns based on values m_i, as the total supply a_i is not equal the total demand b_j when i=1,2,3,4,5 and j=1,2,3,4,5,6,7 meaning that, the three problems are unbalanced, as they were balanced by adding an imaginary column with a cost equal to zero, so each problem became composed of five rows and eight columns, and the total supply equals the total demand and equals 15, meaning $a_i = b_j = 15$, The basic initial solution was found using Vogel's method. We also encountered during the solution a Degeneracy phenomenon, as the number of occupied squares was is not equal to m+n-1. The Degeneracy phenomenon was treated by adding a value of ∈ to the lowest cost and then adopting the modified distribution method to find the optimal solution. The following tables show the results of the optimal solution (using the Modified Distribution Method), After addressing the phenomenon of Degeneracy that we encountered when The basic acceptable solution using Vogel's method was found.

5.1 The First Studied Problem

	$v_1=1$	$v_2=3$	$v_3=4$	$v_4=6$	$v_5=7$	$v_6=10$	$v_7=12$	$v_8=-1$	
$u_1=0$	2	3	4 1	6 1	7 1^{+}	10	12	0 $\in^{-}$	3
$u_2=0$	1 2	3	4 1	7	9	14	15	0	3
$u_3=-1$	1	2 2	4	5 1	8	10	15	0	3
$u_4=-1$	1	2	3	5	6 1^{-}	9 2	11 +	0	3
$u_5=-1$	3	4	5	7	8	11	13 2^{-}	0 1^{+}	3
	2	2	2	2	2	2	2	1	

2	3	4 1	6 1	7 1	10	12	0	3
1 2	3	4 1	7	9	14	15	0	3
1	2 2	4	5 1	8	10	15	0	3
1	2	3	5	6 1	9 2	11 ∈	0	3
3	4	5	7	8	11	13 2	0 1	3
2	2	2	2	2	2	2	1	

T.C.=4*1+6*1+7*1+1*2+4*1+2*2+5*1+6*1+9*2+11* ∈+13*2+0*1=82+11∈

5.2 The second studied problem

	$v_1=1$	$v_2=5$	$v_3=6$	$v_4=8$	$v_5=12$	$v_6=14$	$v_7=15$	$v_8=0$	
$u_1=0$	1 2	4	6 1^{+}	11	12	13	15	0 $\in^{-}$	3
$u_2=1$	1	2 +	5 1^{-}	7 2	12	13	14	0	3
$u_3=0$	3	5	10	11	12	14 1	15 1	0 1^{+}	3
$u_4=-1$	2	4	9	10	11 2	13 1	14	0	3
$u_5=-2$	1	3 2^{-}	8	9	10	12	13 1^{+}	0	3
	2	2	2	2	2	2	2	1	

1 2	4	6 1	11	12	13	15	0	3
1	2 ∈	5 1	7 2	12	13	14	0	3
3	5	10	11	12	14 1	15 1	0 1	3
2	4	9	10	11 2	13 1	14	0	3
1	3 2	8	9	10	12	13 1	0	3
2	2	2	2	2	2	2	1	

T.C.=1*2+6*1+2*∈+5*1+7*2+14*1+15*1+0*1+11*2+13*1+3*2+13*1=110+2∈

5.3 The third studied problem

4	5	6 2	8	9 1	12	14	0	3
1 2	6	7	8	10	11 1	14	0	3
5	6	7	9	10 1	13	15 1	0 1	3
2	7	8	9 1	11	12 1	15 1	0	3
2	3 2	6	8 1	13	14	15	0	3
2	2	2	2	2	2	2	1	

T.C.=6*2+9*1+1*2+11*1+10*1+15*1+0*1+9*1+12*1+15*1+3*2+8*1=109

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6. GEOMETRIC METHOD FOR FINDING THE SPECTRUM

Let PG(p-1,q) be a projective space of dimension (p-1) over a field F_q and ʅ-flat is a ʅ-dimensional projective subspace of PG(p-1,q), such that $|PG(ʅ,q)|=q^{ʅ}+q^{(ʅ-1)}+q^{(ʅ-2)}+\cdots+q+1$. Now, let $\mathcal{F}_{ʅ}$ be the set of all ʅ-flats in PG(p-1,q), where PG(p-1,q)∋k:i-point⇔k has multiplicity in the columns of a generator matrix G of a code Ꞇ which is a multiset of k points in PG(p-1,q) and denoted by $M_{Ꞇ}$. Now, for any S in PG(p-1,q), $M_{Ꞇ}(S)$ is the multiset {k∈$M_{Ꞇ}$ | k∈S}, the multiplicity of S denoted by $m_{Ꞇ}(S)$, is defined as $m_{Ꞇ}(S)=|M_{Ꞇ}(S)|=\sum_{i=1}^{\gamma_0} i|S\cap C_i|$, where $Ꞇ_i$={k∈PG(p-1,q) | k:i-point}, $0 \le ʅ \le \gamma_0$; $\gamma_0=\max\{i|\exists k\text{:i-point in PG(p-1,q)}\}$. Then it holds that $m=m_{Ꞇ}(PG(p-1,q))$, $m-d=\max\{m_{Ꞇ}(\pi)|\pi\in\mathcal{F}_{p-2}\}$. Now, let $a_i=|\{\pi\in\mathcal{F}_{p-2}|m_{Ꞇ}(\pi)=i\}|$, the list of a_i's is the spectrum of a code Ꞇ:$[m,p,d]_q$ with $a_{m-d}=1$ such that $a_i=0$ for $m-d-q^t < i < m-d$ for some t∈N. And $a_i=A_{m-i}/(q-1)$ for $0 \le i \le m-d$. We take a code Ꞇ: $[20,4,10]_2$ as an example of PG(3,2) with generator matrix:

$$G=\begin{bmatrix} 0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\ 1&1&0&0&1&1&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\ 0&1&0&1&0&1&0&1&0&0&1&1&0&0&1&1&0&0&1&1 \end{bmatrix}$$

Such that,

1. $a_i = 0$ for $m-d-q^t < i < m-d$; $t \in N$.
2. $a_{s\theta_{p-2}} = \theta_{p-1}$; s∈N.
3. $a_i = Ꞇ_{m-i} / q-1$, $0 \le i \le m-d$.

So that, $a_0 = A_{20} / 2-1= A_{20} = 0$, $a_1 = A_{19} = 0$,…., $a_6 = A_{14}=1$, $a_7 = A_{13} = 0$, $a_8 = A_{12} = 3$, $a_9 = 0$ from (1), $a_{10} = A_{10} = 11$. Then we get the W.D. $0^1,10^{11},12^3, 14^1$ with spectrum: $(a_6, a_8, a_{10}) = (1,3,11)$.

6.1 Construction the code Ꞇ from m=5

We take m=5 which represents the optimal value of the code Ꞇ:[5,4] over F_2. Such that, the generator matrix is: $G=\begin{bmatrix}1&0&0&0&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\end{bmatrix}$; with codewords: C.W.={$[w_i*G]_q{}^{p}{}_{*m})$ | w_i ∈ PG(p-1,q)}= G^*_{16*5} and d=1 then, we get Ꞇ:$[5,4,1]_2$-code. Where:

1. $a_i = 0$ for $m-d-q^t < i < m-d$, $t \in N$.
2. $a_{s\theta_{p-2}} = \theta_{p-1}$; s∈N.
3. $a_i = Ꞇ_{m-i} / q-1$, $0 \le i \le m-d$.
4. $a_i = (a_{m-d-(p-1)}, a_{m-d-(p-2)},\ldots, a_{m-d})$.
5. $\sum a_i = \sum Ꞇ_i = q^3 + q^2 + q + 1$.

From the above we get: $a_0 = Ꞇ_5=1$, $a_1 = Ꞇ_4=1$, $a_2 = Ꞇ_3=6$, $a_3 = Ꞇ_2 = 6$, $a_4=Ꞇ_1 = 1$, $a_5 = 0$.
Then the W.D.: $0^1, 1^1,2^6, 3^6, 4^1, 5^1$ with spectrum: $(a_0, a_1, a_2, a_3, a_4) = (1,1,6,6,1)$.

6.2 Construction the code Ꞇ from m=7

we take m=7 which represents an optimal Value for the code Ꞇ:[7,4] over F_2. Such that, the generator matrix is: $G=\begin{bmatrix}1&1&1&1&1&1&1\\1&1&1&1&0&0&0\\1&1&0&0&1&1&0\\1&0&1&0&1&0&1\end{bmatrix}$; with codewords: C.W.= G^*_{16*7} and d=3, then we get Ꞇ:$[7,4,3]_2$ – code. Where: $a_0= Ꞇ_7 = 1$, $a_3 = Ꞇ_4 = 7$, $a_4 = Ꞇ_3 = 7$. Then we get W.D.: $0^1, 3^7, 4^7, 7^1$ with spectrum: $(a_0, a_3, a_4) = (1,7,7)$.

6.3 Construction the code Ꞇ from m=15

We take m=15 which represents an optimal Value for the code Ꞇ: [15,4] over F_2. Such that, the generator matrix is:

$$G=\begin{bmatrix}1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&0&0&0&0&1&1&1&1&0&0&0\\1&1&0&0&1&1&0&0&1&1&0&0&1&1&0\\1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\end{bmatrix};$$

with codewords: C.W. = G^*_{16*15} and d=8, then we get Ꞇ:$[15,4,8]_2$-code. Where: $a_0 = Ꞇ_{15} = 0$, $a_1 = a_2 = a_3 = a_4 = \cdots = a_6 = 0$ and $a_7 = a_8 = 15$. Then we get W.D.: $0^1, 8^{15}$ with spectrum: $a_7=15$.

7. CONCLUSIONS AND FUTURE WORKS

From the above results, we used a new geometric method for finding the spectrum and weight distribution of a linear code $[m_i, p, d]$ over a field F_2 in PG(3,2) such that, $a_i=0$ for $m-d-q^t < i < m-d$; t ∈ N. And $a_i=A_{m-i}/(q-1)$ for $0 \le i \le m-d$. we construct a new linear codes of dimension p=4 and smallest length m_i for which a $[m_i, p, d]$-codes, as shown in the table below:

I	Ꞇ: $[m,4,d]_2$	W.D.	Spectrum
1	Ꞇ: $[20,4,10]_2$	$0^1,10^{11},12^3, 14^1$	(1,3,11)
2	Ꞇ: $[5,4,1]_2$	$0^1, 1^1,2^6, 3^6, 4^1, 5^1$	(1,1,6,6,1)
3	Ꞇ: $[7,4,3]_2$	$0^1, 3^7, 4^7, 7^1$	(1,7,7)
4	Ꞇ: $[15,4,8]_2$	$0^1, 8^{15}$	15

In the future, we recommend finding the spectrum and weight distribution of linear codes $[m_i, p, d]$ over F_3 in PG (3,3) in a different way.

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