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SOLUTIONS OF LINEAR MULTI-DIMENSIONAL FRACTIONAL ORDER VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT

In this paper, the aim studying this topic is to extend the study of the one-dimensional fractional to the multi-dimensional fractional integral equations and their applications. The multi-dimensional Laplace transform method (M.D.L.T.M) is developed to solve multi-dimensional fractional Integrals equations. We used the one-dimensional Laplace transform for solving the fractional integral. The procedure will simply to find the Laplace transform to the equation, to solve the transform of the unknown function. Finally, find the inverse Laplace to obtain our desired solution. The result reveals that the transform method is very convenient and effective.
Keywords: *Fractional integral equations and Laplace transform method.*

1. INTRODUCTION

It is known that integrals of integer-order have clear geometric and physical interpretations, which significantly simplify their use for solving applied problems in various fields of science. In the introduction, the idea that the formulation of the concept of fractional integrals was a natural outgrowth of integer-order integrals in much the same way that the fractional exponent follows from the more traditional integer-order exponent has briefly been discussed [2]. The systems of integral equations appear in scientific applications in engineering, chemistry, physics and populations growth models. Studies of systems of integral equations have attracted much concern in applied sciences [1] and [7] the intuitive idea of fractional order calculus is as old as integer order calculus. It can be observed from a letter that written by Leibniz to L'Hospital. The fractional order calculus is a generalization of the integer order calculus to a real or complex number. The authors implemented one dimensional Laplace transform method as an alternative method for solving certain Volterra singular fractional integral equations. [12] discusses the method of integration by partial-fraction decomposition [11]. A solution to the problem geometric and physical interpretation for fractional integration is proposed for Riemann-Liouville fractional integration. The geometric and physical interpretation of Volterra integral of convolution type is then similar to the proposed interpretations for fractional integrals. [13] used Laplace transform to solve some linear fractional differential and integral equations. The procedure will be simple by finding the Laplace transform of the equation, solving the transform of unknown function. Finally, finding the inverse Laplace to btain our desired solution. [6] gave the idea of Volterra fractional integrals with their basic properties. Several methods including the Laplace transform are discussed in introducing Riemann-Liouville fractional integral to solve integral equation. By different ways, fractional integral idea can be described. Now by considering homogenous n-th order ordinary differential equation (initial value problem),

$$D^n y = 0, y^{(k)}(a) = 0, 0 \leq K \leq n-1$$

The solution is the fundamental set $1, x, x^2, \ldots, x^{n-1}$,

$$y = \sum_{n-1}^{n-1} c_r x^r.$$

[5] introduce partial fractions method allowing one to simplify proper rational functions of P(s)/Q(s) and this method is very helpful for

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solving integrals and finding the inverse Laplace Transform of polynomial functions. [20] gave some applications of fractional calculus in Engineering, Abdul-Majid W. (2011) used Laplace transform method as important method with his properties for solving integral equations with Volterra type of the second kind, and gave some applications by using the linearity and the convolution properties to find the exact solution and to support the study.

[9] used method of the Haar wavelet approximating to reduce the fractional Volterra and Abel integral equations to a system of algebraic equations and applied the Haar wavelet operational matrix to fractional integration. The global error bound, some numerical examples with smooth, non-smooth and singular solutions are included to demonstrate the validity and applicability of the method, [4] they established the double Laplace transform formulas for the partial fractional integrals. Then they found it in the sense of Caputo.

[15] introduced the k-fractional integral of the Riemann-Liouville to introduce a fractional integral operator of Weyl type and studied to express this integral operator as certain convolution with the singular kernel

of Riemann-Liouville. [18] gave the fractional integral inequity and singularity with its application, then proved an integral inequality with singularity. [19] derived a solution of a generalized fractional Volterra type integral equation involving K_{4}

function with help of Sumudu transform. The investigation of Sumudu transformation for certain distribution spaces with regard to the fractional integral by [8]. The representation of fractional Riemann-Liouville integral by using the Mikusinski operators have been presented by [16] discussed Mikusinski operators to study fractional

Riemann-Liouville integral operator. [3] developed a method for finding formal solution of certain Fredholm fractional singular integral equations of second kind, analytic solution of the time fractional heat equation and system of partial fractional differential equations. [10] developed a new method is triangular functions to solve linear or non-linear Volterra integral equations where Kernel is weakly singular. [17] introduced two-dimensional hat functions and derive operational matrix of integration of these functions and utilized them to solve some classes of integral equations.

2. DEFINITION OF THE RIEMANN LIOUVILLE FRACTIONAL INTEGRAL, [14]:-

Let v be a real non-negative number. Let f be piecewise continuous on J'=0, and integrable on any finite subinterval of J=[0,∞]. Then for t>0, is the Riemann-Liouville fractional integral of f of order v.

$$ {}_cD_x^{-v} = \frac{1}{\Gamma(v)}\int_c^x (x-t)^{v-1}f(t)dt, v>0. $$

Where f is a known function of t.

Example (1):-

Let's evaluate where Re v>0, μ>-1.

Solution:-

By definition

$$ \begin{aligned} D^{-v}x^{\mu} &= \frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1}t^{\mu}dt \\ &= \frac{1}{\Gamma(v)}\int_0^x (1-\frac{t}{x})^{v-1}x^{v-1}t^{\mu}dt \\ &= \frac{1}{\Gamma(v)}\int_0^1 (1-u)^{v-1}x^{v-1}(xu)^{\mu}xdu \\ &= \frac{1}{\Gamma(v)}x^{\mu+v}\int_0^1 u^{\mu}(1-u)^{v-1}du \\ &= \frac{1}{\Gamma(v)}x^{\mu+v}B(\mu+1,v) \\ &= \frac{\Gamma(\mu)+1}{\Gamma(\mu+v+1)}x^{\mu+v}. \end{aligned} $$

In the above example, we have established that:-

$$ D^{-v}x^{\mu} = \frac{\Gamma(\mu)+1}{\Gamma(\mu+v+1)}x^{\mu+v}, v>0, \mu>-1, x>0. $$

is the Power Rule.

2.1 The One-Dimensional Fractional Integral of f of Order v, [13].

Consider the fractional order for one-dimensional Integral equation with constant coefficient:

$$ {}_cD_x^{-v} = \frac{1}{\Gamma(v)}\int_c^x (x-t)^{v-1}f(t)dt, v>0. $$

We will consider one approach uses the theory of linear order differential equations.

Consider the n-th order differential equation together with the initial conditions:-

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$$y^{n}(x)=f(x), y(c)=0, y'(c)=0,\dots,y^{n-1}(c)=0.$$

By using a form of the Cauchy function,

$$H(x,t)=\frac{(x-t)^{n-1}}{(n-1)!}$$

The unique solution of the above equation is given by:-

$$y(x)=\int_c^x \frac{(x-t)^{n-1}}{(n-1)!}f(t)dt.$$

By using the mathematical induction:-

If n=1, then:-

$$y'(x)=f(x), y(c)=0.$$

We obtain

$$\int_c^x y'(t)dt=\int_c^x \frac{(x-t)^{1-1}}{(1-1)!}f(t)dt.$$

Since y(c) = 0, we have:-

$$y(x)=\int_c^x f(t)dt.$$

If n=n+1, then:-

$$y^{n+1}(x)=f(x), y(c)=0, y'(c)=0,\dots,y^{n}(c)=0.$$

Since

$$y^{n+1}(x)=y^{(')^n}(x)$$

Let $u(x)=y'(x)$, then:-

$u^n(x)=f(x)$, $u(c)=0, u'(c)=0,\dots,u^{(n-1)}(c)=0$.

By using the induction hypothesis, we observe that,

$$\begin{aligned}\int_c^x y'(t)dt &= \int_{z=c}^x\left(\int_{t=c}^z \frac{(z-t)^{n-1}}{(n-1)!}f(t)dt\right)dz\\ y(x)-y(c) &= \int_{t=c}^x\left(\int_{z=t}^x \frac{(z-t)^{n-1}}{(n-1)!}f(t)dz\right)dt\\ &= \int_c^x \frac{(x-t)^n}{(n)!}f(t)dt.\end{aligned}$$

Since

y(c)=0, then:-

$$y(x)=\int_c^x \frac{(x-t)^n}{(n)!}f(t)dt.$$

It is known that the one-dimensional Laplace transform method is an important method that can used to solve the fractional order for one-dimensional integral equation.

2.2 Solution of the Linear Fractional Volterra Integral Equations Via the Laplace Transform, [13]:-

Consider the following one-dimensional linear fractional Volterra integral equation of the second kind:-

$$D^{-v}y(x)=I^{-v}y(x)=\frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1}f(t)dt.$$

Where f is a known function of t and y is the unknown function that must be determined.

Then, taking the Laplace transform to above equation and using the linearity and the convolution properties, one can obtain:

$$L\{D^{-v}y(x)\}=\frac{1}{\Gamma(v)}L\{t^{v-1}\}L\{f(t)\}=s^{-v}Y(s).$$

Where

$L\{y(x)\}=Y(s)$.

Now, taking the inverse one-dimensional Laplace transform to above equation, one can obtain:

$D^{-v}y(x)=L\{s^{-v}y(s)\}$

is the solution of the above Volterra fractional integral equation.

To illustrate this approach, consider the following example.

Example (2):-

Consider the following one-dimensional linear Volterra fractional integral equation of the second kind:

$$I^{-\left(\frac{1}{2}\right)}y(x)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-t)^{\left(\frac{1}{2}\right)-1}e^t\left(\frac{1}{\sqrt{\pi t}}-2e^{4t}erfc(2\sqrt{t})\right)dt$$

where v>0.

Then, taking the Laplace transform to above equation and using the linearity and convolution properties, one can obtain:

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$$L\left\{I^{-(\frac{1}{2})}y(x)\right\} = \frac{4-1}{\sqrt{s}(s-1)(\sqrt{s}+2)}.$$

Now, taking the inverse one-dimensional Laplace transform to the above equation, one can obtain:

$$I^{-(\frac{1}{2})}y(x) = e^{t}[2erf(\sqrt{t})-1] + e^{4t}erfc(2\sqrt{t})$$

is the solution of the above Volterra fractional integral equation.

Example (3):-

Consider the one-dimensional Volterra fractional integral equation:

$$I^{-(\frac{1}{2})}y(x) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{(\frac{1}{2})-1}e^{a^2t}erfc(a\sqrt{t})e^{\frac{-k^2}{4t}}\frac{k}{2\sqrt{\pi t^3}}dt,$$

where v>0.

Then, taking the Laplace transform to above equation and using the linearity and convolution properties, one can obtain:

$$L\left\{I^{-(\frac{1}{2})}y(x)\right\} = \frac{e^{-k\sqrt{s}}}{s^{\frac{1}{2}}(\sqrt{s}(\sqrt{s}+a))}, v>0.$$

Now, taking the inverse Laplace transform to above equation, one can obtain:

$$I^{-v}y(x) = -e^{a^2t}e^{ak}erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right) + erfc\left(\frac{k}{2\sqrt{t}}\right)$$

is the solution of the above Volterra fractional integral equation.

Example (4):-

Consider the one-dimensional linear Volterra fractional integral equation of the second kind:

$$I^{-v}y(x) = \frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1}sin(t)dt,$$

Then, taking the Laplace transform to above equation and using the linearity and convolution properties, one can obtain:

$$L\left\{I^{-v}y(t)\right\} = \frac{1}{s^v(s^2+1)} = \frac{1}{s^v(s+1)(s-1)}, v>0.$$

Now, taking the inverse Laplace transform to above equation, one can obtain:

$$I^{-v}y(x) = L^{-1}\left\{\frac{1}{s^v(s+1)(s-1)}\right\} = \frac{t^{v-1}}{\Gamma(v)}\left\{\frac{e^{-t}-e^{t}}{-2}\right\},$$

is the solution of the above Volterra fractional integral equation.

Example (5):-

Consider the one-dimensional fractional order integral equation:

$$I^{-(\frac{1}{2})}y(x) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{(\frac{1}{2})-1}\frac{1}{\sqrt{\pi t}}\cdot\left(\frac{1}{\sqrt{\pi t}} - \frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}} e^{\tau^2}d\tau\right)dt$$

where v>0.

Then, by taking the Laplace transform to above equation and using the linearity and convolution properties, one can obtain:

$$L\left\{I^{-(\frac{1}{2})}y(x)\right\} = \frac{1}{\sqrt{s}}\left(\frac{1}{\sqrt{s}}\right)\left(\frac{\sqrt{s}}{s+a^2}\right), v>0.$$

Now, by taking the inverse Laplace transform of above equation, one can obtain:

$$I^{-v}y(x) = L^{-1}\left\{\frac{1}{\sqrt{s}(s+a^2)}\right\} = \frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}} e^{\tau^2}d\tau$$

is the solution of the above Volterra fractional integral equation.

3. DEFINITION OF THE RIEMANN LIOUVILLE MULTI-DIMENSIONAL FRACTIONAL INTEGRAL:-

Let v_i for each i=1,2,...,n be a real non- negative number. Let f be a function of t_1 , t_2 ,…,t_n piecewise continuous on J'=(0,∞) and integrable on any finite subinterval of J=[0,∞]. Then, for t_i>0 for each i=1,2,...,n , is the Riemann-Liouville fractional integral of f of order v_i for each i=1,2,...,n.

$$\prod_{i=1}^{n} D_{x_i}^{-v_i} = \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)}\int_{c_n}^{x_n}\int_{c_{n-1}}^{x_{n-1}}\dots\int_{c_1}^{x_1}\prod_{i=1}^{n}(x_i-t_i)^{v_i-1}$$

$$f(t_1,t_2,...,t_n)dt_1dt_2...dt_n.v_i>0, i=0,1,2,...,n,$$

Where f is a known function of t_1 , t_2 ,…,t_n.

Example (6):-

Let's evaluate

$$\prod_{i=1}^{n} D^{-v_i}x_i^{\mu_i},$$

where Re v_i>0 for each i=1,2,...,n, μ_i>-1 for each i=1,2,...,n.

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Solution:-

By definition

$$\prod_{i=1}^{n} D^{-v_i} x_i^{\mu_i} = \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \int_0^{x_n}\int_0^{x_{n-1}}\dots\int_0^{x_1}\prod_{i=1}^{n}\left(1-\frac{t_i}{x_i}\right)^{v_i-1} x_i^{v_i-1} t_i^{\mu_i}\, dt_1 dt_2 \dots$$

$$= \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \int_0^{1}\int_0^{1}\dots\int_0^{1}\prod_{i=1}^{n}(1-u_i)^{v_i-1}(x_i)^{v_i-1}(xu)_i^{\mu_i} x_i\, du_1 d \dots_n$$

$$= \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \int_0^{1}\int_0^{1}\dots\int_0^{1}\prod_{i=1}^{n}(1-u_i)^{v_i-1}(x_i)^{v_i-1}(xu)_i^{\mu_i} x_i d$$

$$= \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \prod_{i=1}^{n} x_i^{\mu_i+v_i} \int_0^{1}\int_0^{1}\dots\int_0^{1}\prod_{i=1}^{n}(1-u_i)^{v_i-1}(u_i)^{\mu_i} d$$

$$= \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \prod_{i=1}^{n} x_i^{\mu_i+v_i} B_i(\mu_i+1, v_i)$$

$$= \prod_{i=1}^{n}\left\{\frac{\Gamma(\mu_i)+1}{\Gamma(\mu_i+v_i+1)} x^{\mu_i+v_i}\right\}.$$

In the above example, we have established that:-

$$\prod_{i=1}^{n} D^{-v_i} x_i^{\mu_i} = \prod_{i=1}^{n}\left\{\frac{\Gamma(\mu_i)+1}{\Gamma(\mu_i+v_i+1)} x^{\mu_i+v_i}\right\}.$$

is the multi-dimensional Power Rule.

3.1 The Multi-Dimensional Fractional Integral of u of Order v_i, i=0,1,2,...,n.

Consider the fractional order for multi-dimensional Integral equation with constant coefficient:

$$\prod_{i=1}^{n} D^{-v_i} u(x_1, x_2, \dots, x_n) = \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \int_{c_n}^{x_n}\int_{c_{n-1}}^{x_{n-1}}\dots\int_{c_1}^{x_1}\prod_{i=1}^{n}(x_i-t_i)^{v_i-1} u(t_1, t_2, \dots, t_n)\, dt_1 d$$

v_i, i=0,1,2,...,n.

By uses theory of linear partial differential equations. Consider n-th order partial differential equation together with the initial and boundary conditions:-

$$\frac{\partial^{\sum_{i=1}^{n} m_i} u(x_1, x_2, \dots, x_n)}{\prod_{i=1}^{n}\partial x_i^{m_i}} = f(x_1, x_2, \dots, x_n).$$

Using a form of the Cauchy function,

$$H(x_1, x_2, \dots, x_n, t_1, t_2, \dots, t_n) = \prod_{i=1}^{n}\frac{(x_i-t_i)^{n_i-1}}{(n_i-1)!}$$

The unique solution of above equation is given by:-

$$u(x_1, x_2, \dots, x_n) = \int_{c_n}^{x_n}\int_{c_{n-1}}^{x_{n-1}}\dots\int_{c_1}^{x_1}\prod_{i=1}^{n}\frac{(x_i-t_i)^{n_i-1}}{(n_i-1)!}$$

$$\cdot f(t_1, t_2, \dots, t_n)\, dt_1 dt_2 \dots dt_n.$$

$$\int_{c_1}^{x_1}\frac{\partial^{\sum_{i=1}^{n} m_i} u(t_1, t_2, \dots, t_n)}{\partial t_1^{m_1}} dt_1 = \int_{c_n}^{x_n}\int_{c_{n-1}}^{x_{n-1}}\dots\int_{c_1}^{x_1}\prod_{i=2}^{n}\frac{(x_1-t_1)^{1-1}(x_i-t_i)^{n_i-1}}{(1-1)!(n_i-1)!}$$

$$\cdot f(t_1, t_2, \dots, t_n)\, dt_1 dt_2 \dots dt_n.$$

Together with the initial and boundary conditions:-

$$u(x_1, x_2, \dots, x_n) = \int_{c_n}^{x_n}\int_{c_{n-1}}^{x_{n-1}}\dots\int_{c_2}^{x_2}\prod_{i=2}^{n}\frac{(x_i-t_i)^{n_i-1}}{(n_i-1)!}$$

$$\cdot f(t_1, t_2, \dots, t_n)\, dt_1 dt_2 \dots dt_n.$$

3.2 Solution of the Multi-Dimensional Volterra Linear Fractional Integral Equations Via the Multi-Dimensional Laplace Transform

Consider the following linear fractional order for multi-dimensional Integral equation of the second kind:

$$\prod_{i=1}^{n} I_i^{v_i} y(x_1, x_2, \dots, x_n) = \frac{1}{\prod_{i=1}^{n}\Gamma(v_i)} \int_0^{x_n}\int_0^{x_{n-1}}\dots\int_0^{x_1}$$

$$\prod_{i=1}^{n}(x_i-t_i)^{v_i-1} f(t_1, t_2, \dots, t_n)\, dt_1 dt_2 \dots dt_n,$$

$v_i > 0$, i=0,1,2,...,n.

Where f is a known function of t_1, t_2, …, t_n and y is the unknown function that must be determined.

Then, taking the multi-dimensional Laplace transform to both sides of above equation, by using the Linear and convolution properties and by using Gamma function

Gamma (v)=(v-1)! and Gamma(v+1)=v Gamma(v), we have:-

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$$L_n\left\{\prod_{i=1}^{n} I_i^{v_i} y(x_1,x_2,...,x_n)\right\} = \prod_{i=1}^{n} s_i^{-v_i} Y(s_1,s_2,...,s_n).$$

Where

$$L_n\{y(x_1,x_2,...,x_n)\} = Y(s_1,s_2,...,s_n).$$

Now, taking the inverse multi-dimensional Laplace transform to both sides of above equation, one can obtain:

$$\prod_{i=1}^{n} I_i^{v_i} y(x_1,x_2,...,x_n) = L_n^{-1}\left\{\prod_{i=1}^{n} s_i^{-v_i} Y(s_1,s_2,...,s_n)\right\}.$$

is the solution of the above fractional order for multi-dimensional Volterra integral equation.

To illustrate this method, consider the following example.

Example (7):-

Consider linear two-dimensional fractional order integral equation of the second kind:-

$$\prod_{i=1}^{2} I_i^{v_i} y(x_1,x_2) = \frac{1}{\prod_{i=1}^{2}\Gamma(v_i)}\int_0^{x_2}\int_0^{x_1}\prod_{i=1}^{2}(x_i-t_i)^{v_i-1}$$

$$f(t_1,t_2)dt_1dt_2.,$$

$v_i > 0$, i=1,2.

$$I_1^{\frac{5}{3}} I_2^{\frac{7}{2}} y(x_1,x_2) = \frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{7}{2})}\int_0^{x_2}\int_0^{x_1}(x_1-t_1)^{\frac{5}{3}-1}$$

$$(x_2-t_2)^{\frac{7}{2}-1} e^{a(x_1-t_1)+b(x_2-t_2)} dt_1dt_2.$$

Then, taking two-dimensional Laplace Transform to both sides of above equation, using the Linear and convolution properties and by using Gamma function, one can obtain:-

$$L_2\left\{I_1^{\frac{5}{3}} I_2^{\frac{7}{2}} y(x_1,x_2)\right\} = \frac{1}{s_1^{\frac{5}{3}} s_2^{\frac{7}{2}}(s_1-a)^2(s_2-b)^2}$$

Now, taking the inverse two-dimensional Laplace transform to both sides of above equation, one can have:-

$$I_1^{\frac{5}{3}} I_2^{\frac{7}{2}}\{U(s_1,s_2)\} = L_2^{-1}\left\{s_1^{\frac{-5}{3}} s_2^{\frac{-7}{2}}\frac{1}{(s_1-a)^2(s_2-b)^2}\right\}$$

$$= t_1^{\frac{5}{3}} E_{1,\frac{8}{3}}(at_1) t_2^{\frac{7}{2}} E_{1,\frac{9}{2}}\left\{(a-b)(e^{at_1}-e^{bt_2})\right\}$$

is the solution of the above fractional Volterra integral equation.

Example (8):-

Consider the linear two-dimensional fractional Volterra integral equation of the second kind:-

$$\prod_{i=1}^{2} I_i^{v_i} y(x_1,x_2) = \frac{1}{\prod_{i=1}^{2}\Gamma(v_i)}\int_0^{x_2}\int_0^{x_1}\prod_{i=1}^{2}(x_i-t_i)^{v_i-1}$$

$$f(t_1,t_2)dt_1dt_2.,$$

$v_i > 0$, i=1,2.

$$I_1^{v_1} I_2^{v_2} x_1^{\frac{1}{2}} x_2^{\frac{2}{3}} = \frac{1}{\prod_{i=1}^{2}\Gamma(v_i)}\int_0^{x_2}\int_0^{x_1}\prod_{i=1}^{2}(x_i-t_i)^{v_i-1}$$

$$t_1^{\frac{1}{2}} t_2^{\frac{2}{3}} dt_1dt_2.$$

Then, taking two-dimensional Laplace transform to both sides of above equation, using the Linear and convolution properties and by using Gamma function, one can obtain:-

$$L_2\left\{I_1^{v_1} I_2^{v_2} x_1^{\frac{1}{2}} x_2^{\frac{2}{3}}\right\} = \left\{\frac{\Gamma(\frac{1}{2}+1)}{s_1^{\frac{1}{2}+v_1+1}}\right\}\left\{\frac{\Gamma(\frac{2}{3}+1)}{s_2^{\frac{2}{3}+v_2+1}}\right\}.$$

Therefore

$$\prod_{i=1}^{2} I_i^{v_i} U(s_1,s_2) = s_1^{-v_1} s_2^{-v_2}\left\{\frac{(\frac{1}{2})!}{s_1^{\frac{3}{2}}}\right\}\left\{\frac{(\frac{2}{3})!}{s_2^{\frac{5}{3}}}\right\}$$

Now, taking the inverse two-dimensional Laplace transform to both sides of above equation, one can have:

$$\prod_{i=1}^{2} I_i^{v_i} L_2^{-1}\{U(s_1,s_2)\} = \frac{t_1^{v_1-1}}{\Gamma(v_1)}\frac{t_2^{v_2-1}}{\Gamma(v_2)} t_1^{\frac{1}{2}} t_2^{\frac{2}{3}}$$

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is the solution of the above Volterra integral equation.

Example (9):-

Consider the linear three-dimensional fractional Volterra integral equation of the second kind:-

$$\prod_{i=1}^{3} I_i^{v_i} y(x_1,x_2,x_3) = \frac{1}{\prod_{i=1}^{3}\Gamma(v_i)} \int_0^{x_3}\int_0^{x_2}\int_0^{x_1} \prod_{i=1}^{3}(x_i-t_i)^{v_i-1}$$

$$f(t_1,t_2,t_3)dt_1dt_2dt_3.,$$

$v_i > 0$, i=1,2,3.

$$\prod_{i=1}^{3} I_i^{v_i} y(x_1,x_2,x_3) = \frac{1}{\prod_{i=1}^{3}\Gamma(v_i)} \int_0^{x_3}\int_0^{x_2}\int_0^{x_1} \prod_{i=1}^{3}(x_i-t_i)^{v_i-1}$$

$$\left(\frac{1}{\frac{1}{2}!\frac{5}{2}!\frac{7}{3}!}(t_1^{\frac{1}{2}}t_2^{\frac{5}{2}}t_3^{\frac{7}{3}}) + t_1^{\frac{1}{4}}t_2^{\frac{3}{4}}t_3^{\frac{5}{4}}\right)dt_1dt_2dt_3.$$

Then, taking three-dimensional Laplace transform to both sides of above equation, using the Linear and convolution properties and by using Gamma function, one can obtain:-

$$I_1^{v_1}I_2^{v_2}I_3^{v_3}U(s_1,s_2,s_3) = s_1^{-v_1}s_2^{-v_2}s_3^{-v_3}$$

$$\left\{\frac{1}{\frac{1}{2}!\frac{5}{2}!\frac{7}{3}!}\left\{\frac{\Gamma(\frac{1}{2}+1)\Gamma(\frac{5}{2}+1)\Gamma(\frac{7}{3}+1)}{s_1^{\frac{1}{2}+v_1+1}s_2^{\frac{5}{2}+v_2+1}s_3^{\frac{7}{3}+v_3+1}}\right\}+\right.$$

$$\left.\frac{1}{\frac{1}{4}!\frac{3}{4}!\frac{5}{4}!}\left\{\frac{\Gamma(\frac{1}{4}+1)\Gamma(\frac{3}{4}+1)\Gamma(\frac{5}{4}+1)}{s_1^{\frac{1}{4}+v_1+1}s_2^{\frac{3}{4}+v_2+1}s_3^{\frac{5}{4}+v_3+1}}\right\}\right\}$$

Now, taking the inverse two-dimensional Laplace transform of above equation, one can have:

$$I_1^{v_1}I_2^{v_2}I_3^{v_3}L_3^{-1}\{U(s_1,s_2,s_3)\} =$$

$$L_3^{-1}\left\{s_1^{-v_1}s_2^{-v_2}s_3^{-v_3}\left\{\frac{1}{\frac{1}{2}!\frac{5}{2}!\frac{7}{3}!}\left\{\frac{\Gamma(\frac{1}{2}+1)\Gamma(\frac{5}{2}+1)\Gamma(\frac{7}{3}+1)}{s_1^{\frac{1}{2}+v_1+1}s_2^{\frac{5}{2}+v_2+1}s_3^{\frac{7}{3}+v_3+1}}\right\}+\right.\right.$$

$$\left.\left.\frac{1}{\frac{1}{4}!\frac{3}{4}!\frac{5}{4}!}\left\{\frac{\Gamma(\frac{1}{4}+1)\Gamma(\frac{3}{4}+1)\Gamma(\frac{5}{4}+1)}{s_1^{\frac{1}{4}+v_1+1}s_2^{\frac{3}{4}+v_2+1}s_3^{\frac{5}{4}+v_3+1}}\right\}\right\}\right\} =$$

$$\frac{t_1^{v_1+\frac{3}{2}-1}t_2^{v_2+\frac{7}{2}-1}t_3^{v_3+\frac{10}{3}-1}}{\Gamma(v_1+\frac{3}{2})\Gamma(v_2+\frac{7}{2})\Gamma(v_3+\frac{10}{3})}+$$

$$\frac{t_1^{v_1+\frac{5}{4}-1}t_2^{v_2+\frac{7}{4}-1}t_3^{v_3+\frac{9}{4}-1}}{\Gamma(v_1+\frac{5}{4})\Gamma(v_2+\frac{7}{4})\Gamma(v_3+\frac{9}{4})}.$$

4. CONCLUSION

In the present paper we implemented the multi-dimensional Laplace transforms method for solving multi-dimensional fractional order Integral equations of the second type. It illustrated that, this method is power, effective and reliable tool for the solution of fractional linear multi-dimensional Integral equations. It is accelerates for the convergence. The procedure will be simple by find the multi-dimensional Laplace transform of the equation, solving the transform of unknown function and applied to find an exact solution of fractional multi-dimensional Integral equations. We hope that, it will benefit many researchers in disciplines of applied mathematics, mathematical physics and engineering.

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